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Convergence conditions for splitting iteration methods for non-Hermitian linear systems ☆

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Abstract

Necessary and sufficient convergence conditions are studied for splitting iteration methods for non-Hermitian system of linear equations when the coefficient matrix is non-singular. When this theory is specialized to the generalized saddle-point problem, we obtain convergence theorem for a class of modified accelerated overrelaxation iteration methods, which include the Uzawa and the inexact Uzawa methods as special cases. Moreover, we apply this theory to the two-stage iteration methods for non-Hermitian positive definite linear systems, and obtain sufficient conditions for guaranteeing the convergence of these methods.

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Keywords: Non-Hermitian linear systems; Splitting iteration method; Convergence

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1. Introduction

Consider a large sparse system of linear equations

$$Ax = b, \quad A \in \mathbb{C}^{n\times n} \quad \text{and} \quad b \in \mathbb{C}^n, \tag{1.1}$$

where the coefficient matrix A is assumed to be non-Hermitian and non-singular. Let

$$\mathscr{H}(A) = \frac{1}{2}(A + A^*) \quad \text{and} \quad \mathscr{S}(A) = \frac{1}{2}(A - A^*)$$

be the Hermitian and the skew-Hermitian parts of the matrix A, respectively. Then we have *Hermitian and skew-Hermitian* (HS) splitting

$$A = \mathscr{H}(A) + \mathscr{S}(A)$$

of the matrix A. In general, by splitting the matrix $A \in \mathbb{C}^{n\times n}$ into

$$A = M - N, \tag{1.2}$$

where $M \in \mathbb{C}^{n\times n}$ is non-singular, we can define the splitting iteration scheme

$$x^{(k+1)} = Tx^{(k)} + M^{-1}b, \quad k = 0, 1, 2, \ldots \tag{1.3}$$

for the system of linear equations (1.1), where $T = M^{-1}N$ is the corresponding iteration matrix. This iterative method is convergent if and only if the spectral radius of its iteration matrix T, denoted by $\rho(T)$, is less than one, i.e., $\rho(T) < 1$, see [13,27,9].

There have been many studies about the convergence of the splitting iteration method (1.3), or in other words, the matrix splitting (1.2), when the coefficient matrix $A \in \mathbb{C}^{n\times n}$ is a monotone matrix, an H-matrix, or a Hermitian positive definite (or semidefinite) matrix, and correspondingly, the splitting (1.2) is a regular splitting, a weak regular splitting, or a P-regular splitting. The main tools used to establish these convergence theorems are the theory of non-negative matrix and the properties of Hermitian positive definite matrix. For details, we refer to [13,27,5,9].

For the two-stage iteration method induced by a second splitting $M = F - G$, Bai [2], Cao [14,15], and Bai and Wang [12], etc., have studied its convergence when the matrix $A \in \mathbb{C}^{n\times n}$ is a monotone matrix, or a Hermitian positive definite matrix, and correspondingly, the outer and the inner splittings, $A = M - N$ and $M = F - G$, are weak regular and regular splittings, or symmetric P-regular splittings, respectively. See also [10,11,18]. Here, we remark that the two-stage iteration method was originally presented by Nichols [25] in 1973 and, then, it was further studied and developed extensively in the literature, e.g., Nichols [26], Golub and Overton [19], Golub and Ye [21], Bai and Qiu [8] and Axelsson et al. [1].

When the matrix $A \in \mathbb{C}^{n\times n}$ is only positive definite and non-Hermitian, Bai [5], and Wang and Bai [28] presented several sufficient conditions for guaranteeing the convergence of the single and the two-stage splitting iteration methods. To our knowledge, there are only a few results on this topic, see [22,23,6,28,24].

In this paper, based upon [13, Theorem 5.32] we further present sufficient and necessary convergence conditions for the iterative method (1.3) for solving the non-Hermitian system of linear equations (1.1). Our only assumption on the coefficient matrix $A \in \mathbb{C}^{n\times n}$ is non-singularity and on the splitting $A = M - N$ is positive definiteness of the matrix $M \in \mathbb{C}^{n\times n}$. When this theory is specialized to the generalized saddle-point problem, we obtain sufficient conditions for the convergence of the modified accelerated overrelaxation iteration method [29], which includes the Uzawa and the inexact Uzawa methods as special cases. See also [22,23]. Moreover, we apply this theory to the above-mentioned two-stage iteration methods and obtain sufficient conditions for guaranteeing their convergence.

2. Preliminaries

A matrix $A \in \mathbb{C}^{n\times n}$ is called positive definite (or semidefinite), if for all $x \in \mathbb{C}^n, x \neq 0$, it holds that $\mathrm{Re}(x^*Ax) > 0$ (or $\mathrm{Re}(x^*Ax) \geqslant 0$). Here, $\mathrm{Re}(\cdot)$ denotes the real part of the corresponding complex number. The spectrum of the matrix A is denoted by $\sigma(A)$.

$A = M - N$ is called a splitting of the matrix A if $M \in \mathbb{C}^{n\times n}$ is non-singular. This splitting is called a convergent splitting if $\rho(M^{-1}N) < 1$; it is called a P-regular splitting if $M + N$ is positive definite; a Hermitian splitting if both M and N are Hermitian; and a Hermitian P-regular splitting if M is Hermitian positive definite and N is Hermitian positive semidefinite. Evidently, a Hermitian P-regular splitting is a P-regular splitting [11]. From [13] we know that if $A \in \mathbb{C}^{n\times n}$ is Hermitian and $A = M - N$ is P-regular, then $\rho(M^{-1}N) < 1$ if and only if A is positive definite.

We consider a non-singular matrix $A \in \mathbb{C}^{n\times n}$, and a matrix $T \in \mathbb{C}^{n\times n}$ such that $I - T$ is non-singular. Then there exists a unique pair of matrices M and N, with $M \in \mathbb{C}^{n\times n}$ non-singular, such that $A = M - N$ and $T = M^{-1}N$. For the iteration matrix $T = M^{-1}N$, we use $\mathscr{E}_T$ to denote the set of eigenvectors of the matrix T with, at least, one eigenvector associated with each of its distinct eigenvalues. We remark that if T has a multiple eigenvalue with several linearly independent eigenvectors, then $\mathscr{E}_T$ includes arbitrary one of these eigenvectors.

The following lemma is crucial for our subsequent discussions.

Lemma 2.1 [13]. *Let $A, M \in \mathbb{C}^{n\times n}$ be non-singular and $A = M - N$. Let $T = M^{-1}N$. If A and M satisfy the conditions*

$$x^*Ax \neq 0 \quad \text{and} \quad \frac{x^*(M^*A^{-*}A + N)x}{x^*Ax} > 0 \tag{2.1}$$

for every x in $\mathscr{E}_T$, then $\rho(T) < 1$. Conversely, if $\rho(T) < 1$, then either (2.1) holds or

$$x^*Ax = x^*(M^*A^{-*}A + N)x = 0$$

holds for every eigenvector x of the matrix T.

3. Convergence conditions

In this section, we derive necessary and sufficient convergence conditions for the splitting (1.2), or for the splitting iteration method (1.3). To this end, we first introduce two concepts as follows.

Definition 3.1. Let $A, M \in \mathbb{C}^{n\times n}$ be non-singular, $A = M - N$ be a splitting of the matrix A, and $T = M^{-1}N$. If

$$x^*Ax \neq 0 \quad \text{and} \quad \frac{x^*\mathscr{H}(M)x \cdot x^*\mathscr{H}(A)x - x^*\mathscr{S}(M)x \cdot x^*\mathscr{S}(A)x}{(x^*\mathscr{H}(A)x)^2 - (x^*\mathscr{S}(A)x)^2} > \frac{1}{2} \tag{3.1}$$

hold for any $x \neq 0$, then we call $A = M - N$ a generalized P-regular splitting. If (3.1) holds for all $x \in \mathscr{E}_T$, we call $A = M - N$ a local P-regular splitting.

Evidently, when $A \in \mathbb{C}^{n\times n}$ is Hermitian and non-singular, the splitting $A = M - N$ is a generalized P-regular splitting if and only if it is a P-regular splitting.

The following lemma gives conditions for guaranteeing $x^*Ax \neq 0$ $(\forall x \in \mathscr{E}_T)$.

Lemma 3.1. *Let $A, M \in \mathbb{C}^{n\times n}$ be non-singular and $A = M - N$. Let $T = M^{-1}N$. Then $x^*Ax \neq 0$ $(\forall x \in \mathscr{E}_T)$ holds if and only if $x^*Mx \neq 0$ $(\forall x \in \mathscr{E}_T)$. In particular, if M is positive definite, then $x^*Ax \neq 0$ $(\forall x \in \mathscr{E}_T)$.*

Proof. Let $\lambda \in \mathbb{C}$ be an eigenvalue and $x \in \mathscr{E}_T$ be a corresponding eigenvector of the matrix T. Then we have $Tx = \lambda x$, or in other words,

$$Ax = (1-\lambda)Mx. \tag{3.2}$$

By premultiplying this equality with x^* on both sides, we get

$$x^*Ax = (1-\lambda)x^*Mx. \tag{3.3}$$

Because $A \in \mathbb{C}^{n\times n}$ is non-singular and $x \in \mathbb{C}^n$ is non-zero, we know that $Ax \neq 0$. Therefore, (3.2) implies that $\lambda \neq 1$ and $Mx \neq 0$. Now, it is straightforward from (3.3) that $x^*Ax \neq 0$ $(\forall x \in \mathscr{E}_T)$ holds if and only if $x^*Mx \neq 0$ $(\forall x \in \mathscr{E}_T)$.

In particular, when $M \in \mathbb{C}^{n\times n}$ is positive definite, it holds that $x^*Mx \neq 0$ $(\forall x \in \mathbb{C}^n \setminus \{0\})$ and $\lambda \neq 1$ due to the non-singularity of the matrix A; see (3.2). Hence, $x^*Ax \neq 0$ $(\forall x \in \mathscr{E}_T)$ follows directly from (3.3). □

Based on Lemma 3.1, we are now ready to demonstrate necessary and sufficient conditions for the convergence of the matrix splitting (1.2).

Theorem 3.1. *Let $A, M \in \mathbb{C}^{n\times n}$ be non-singular and $A = M - N$. Denote by $T = M^{-1}N$ and assume $x^*Mx \neq 0$ $(\forall x \in \mathscr{E}_T)$. Then $\rho(T) < 1$ if and only if $A = M - N$ is a local P-regular splitting.*

Proof. Since $x^*Mx \neq 0$ $(\forall x \in \mathscr{E}_T)$, by Lemma 3.1 we see that $x^*Ax \neq 0$ $(\forall x \in \mathscr{E}_T)$ and, therefore, $x^*A^*x \neq 0$ $(\forall x \in \mathscr{E}_T)$. In addition, by Lemma 2.1 we know that $\rho(T) < 1$ if and only if

$$\frac{x^*(M^*A^{-*}A + N)x}{x^*Ax} > 0 \quad \forall x \in \mathscr{E}_T.$$

Let λ be an eigenvalue of the matrix T and $x \in \mathscr{E}_T$ be a corresponding eigenvector, i.e., $Tx = \lambda x$. Then it holds that

$$Nx = \lambda Mx \quad \text{and} \quad Ax = (1-\lambda)Mx.$$

Moreover, from the proof of Lemma 3.1 we see that $\lambda \neq 1$. It then follows from straightforward derivations that

$$\begin{aligned}\frac{x^*(M^*A^{-*}A + N)x}{x^*Ax} &= \frac{(Mx)^*A^{-*}Ax + x^*Nx}{x^*Ax}\\ &= \frac{\left(\frac{1}{1-\lambda}Ax\right)^* A^{-*}Ax + x^*Nx}{x^*Ax}\\ &= \frac{\left(\frac{1}{1-\lambda}x\right)^* Ax + x^*Nx}{x^*Ax}\\ &= \frac{\left(\frac{1}{1-\frac{x^*Nx}{x^*Mx}}\right)^* x^*Ax + x^*Nx}{x^*Ax}\end{aligned}$$

$$
\begin{aligned}
&= \frac{\left(\frac{x^*Ax}{1-\frac{x^*N^*x}{x^*M^*x}} + x^*Nx\right)\cdot x^*A^*x}{x^*Ax\cdot x^*A^*x} \\
&= \frac{x^*Ax\cdot x^*M^*x + x^*Nx\cdot x^*A^*x}{x^*Ax\cdot x^*A^*x} \\
&= \frac{x^*Ax\cdot x^*M^*x + x^*Mx\cdot x^*A^*x - x^*Ax\cdot x^*A^*x}{x^*Ax\cdot x^*A^*x} \\
&> 0.
\end{aligned}
$$

This inequality is obviously equivalent to

$$\frac{x^*\mathscr{H}(M)x\cdot x^*\mathscr{H}(A)x - x^*\mathscr{S}(M)x\cdot x^*\mathscr{S}(A)x}{(x^*\mathscr{H}(A)x)^2-(x^*\mathscr{S}(A)x)^2} > \frac{1}{2} \quad \forall x\in\mathscr{E}_T,$$

or that $A = M - N$ is a local P-regular splitting of the matrix $A\in\mathbb{C}^{n\times n}$. $\square$

We remark that the conditions in Theorem 3.1 might be more easily checked than those in Lemma 2.1, since the former does not involve A^{-1}. Note that under the assumptions of Theorem 3.1, the inequality in (3.1) can be rewritten as

$$\frac{x^*\mathscr{H}(M)x}{x^*\mathscr{H}(A)x}\cdot\frac{1-\frac{x^*\mathscr{S}(M)x}{x^*\mathscr{H}(M)x}\cdot\frac{x^*\mathscr{S}(A)x}{x^*\mathscr{H}(A)x}}{1-\left(\frac{x^*\mathscr{S}(A)x}{x^*\mathscr{H}(A)x}\right)^2} > \frac{1}{2}.$$

This shows that the convergence of the splitting $A = M - N$ closely depends on the quantities

$$\frac{x^*\mathscr{H}(M)x}{x^*\mathscr{H}(A)x},\quad \frac{x^*\mathscr{S}(M)x}{x^*\mathscr{H}(M)x}\quad\text{and}\quad\frac{x^*\mathscr{S}(A)x}{x^*\mathscr{H}(A)x},$$

which are associated with the generalized eigenvalues of the matrix pencils $(\mathscr{H}(M),\mathscr{H}(A))$, $(\mathscr{S}(M),\mathscr{H}(M))$ and $(\mathscr{S}(A),\mathscr{H}(A))$. Note that $\mathscr{H}(A)$, $\mathscr{S}(A)$ and $\mathscr{H}(M)$, $\mathscr{S}(M)$ are normal matrices.

Theorem 3.1 immediately yields the following corollaries.

Corollary 3.1. *Let $A, M\in\mathbb{C}^{n\times n}$ be non-singular and $A = M - N$. Denote by $T = M^{-1}N$ and $P = 2M - A$, and assume $x^*Mx \neq 0$ $(\forall x\in\mathscr{E}_T)$. Then $\rho(T) < 1$ if and only if*

$$x^*\mathscr{H}(P)x\cdot x^*\mathscr{H}(A)x - x^*\mathscr{S}(P)x\cdot x^*\mathscr{S}(A)x > 0 \quad \forall x\in\mathscr{E}_T. \tag{3.4}$$

Proof. By Lemma 3.1 we know that $x^*Mx \neq 0$ $(\forall x\in\mathscr{E}_T)$ if and only if $x^*Ax\neq 0$ $(\forall x\in\mathscr{E}_T)$. Therefore,

$$(x^*\mathscr{H}(A)x)^2-(x^*\mathscr{S}(A)x)^2 > 0 \quad \forall x\in\mathscr{E}_T.$$

It then follows that the local P-regularity of the splitting $A = M - N$ is equivalent to (3.4). By Theorem 3.1 we know that the conclusion of this corollary is true. $\square$

Corollary 3.2. *Let $A, M\in\mathbb{C}^{n\times n}$ be positive definite and $A = M - N$. Denote by $T = M^{-1}N$. Then $\rho(T) < 1$ if and only if $A = M - N$ is a local P-regular splitting.*

Corollary 3.3. *Let $A\in\mathbb{C}^{n\times n}$ be positive semidefinite and non-singular, and $M\in\mathbb{C}^{n\times n}$ be positive definite. Let $A = M - N$ and $T = M^{-1}N$. Then $\rho(T) < 1$ if and only if $A = M - N$ is a local P-regular splitting.*

Corollary 3.4. *Let $A \in \mathbb{C}^{n\times n}$ be positive semidefinite and non-singular, and $M \in \mathbb{C}^{n\times n}$ be positive definite. Let $A = M - N$ and $T = M^{-1}N$. Then $\rho(T) < 1$, provided $P := 2M - A$ is positive definite and the skew-Hermitian part $\mathscr{S}(P)$ of the matrix P is equal to $\tau\mathscr{S}(A)$ for some $\tau > 0$.*

Proof. By Corollary 3.3 we know that $\rho(T) < 1$ if and only if

$$\frac{x^*\mathscr{H}(M)x \cdot x^*\mathscr{H}(A)x - x^*\mathscr{S}(M)x \cdot x^*\mathscr{S}(A)x}{(x^*\mathscr{H}(A)x)^2 - (x^*\mathscr{S}(A)x)^2} > \frac{1}{2} \quad \forall x \in \mathscr{E}_T. \tag{3.5}$$

Because $M \in \mathbb{C}^{n\times n}$ is positive definite, Lemma 3.1 shows that $x^*Ax \neq 0$ holds for $\forall x \in \mathscr{E}_T$. Hence,

$$(x^*\mathscr{H}(A)x)^2 - (x^*\mathscr{S}(A)x)^2 > 0 \quad \forall x \in \mathscr{E}_T.$$

We can now rewrite the inequality (3.5) as

$$x^*(2\mathscr{H}(M) - \mathscr{H}(A))x \cdot x^*\mathscr{H}(A)x - x^*(2\mathscr{S}(M) - \mathscr{S}(A))x \cdot x^*\mathscr{S}(A)x > 0 \quad \forall x \in \mathscr{E}_T,$$

or equivalently,

$$x^*(\mathscr{H}(2M - A))x \cdot x^*\mathscr{H}(A)x - x^*(\mathscr{S}(2M - A))x \cdot x^*\mathscr{S}(A)x > 0 \quad \forall x \in \mathscr{E}_T.$$

Therefore, (3.5) is true if and only if

$$x^*\mathscr{H}(P)x \cdot x^*\mathscr{H}(A)x - x^*\mathscr{S}(P)x \cdot x^*\mathscr{S}(A)x > 0 \quad \forall x \in \mathscr{E}_T \tag{3.6}$$

is true. Evidently, a sufficient condition for guaranteeing the validity of (3.6) is that P is positive definite and $\mathscr{S}(P) = \tau\mathscr{S}(A)$ for some $\tau > 0$. $\square$

The following splitting introduced in [23,29] is a concrete example that satisfies the conditions in Corollary 3.4 and is, hence, a convergent splitting. See also [22].

Example 3.1. Let $A \in \mathbb{C}^{n\times n}$ be positive semidefinite and non-singular, and the diagonal entries of its skew-Hermitian part $\mathscr{S}(A)$ are zero. Define a splitting of the matrix $A \in \mathbb{C}^{n\times n}$ as

$$A = M(\omega,\tau) - N(\omega,\tau), \tag{3.7}$$

with

$$M(\omega,\tau) = \frac{1}{\tau}(B_c + \omega K_{\mathrm{L}}) \quad \text{and} \quad N(\omega,\tau) = \frac{1}{\tau}(B_c + \omega K_{\mathrm{L}} - \tau A),$$

where $K_{\mathrm{L}} \in \mathbb{C}^{n\times n}$ is the strictly lower triangular part of $\mathscr{S}(A)$, $B_c \in \mathbb{C}^{n\times n}$ is some prescribed Hermitian positive definite matrix such that the matrix $P(\omega,\tau) := 2M(\omega,\tau) - A$ is positive definite, and $\omega > \tau > 0$.

We notice that

$$\mathscr{H}(M(\omega,\tau)) = \frac{1}{\tau}\left(B_c + \frac{\omega}{2}(K_{\mathrm{L}} + K_{\mathrm{L}}^*)\right)$$

and

$$\mathscr{S}(M(\omega,\tau)) = \frac{\omega}{2\tau}(K_{\mathrm{L}} - K_{\mathrm{L}}^*) = \frac{\omega}{\tau}\mathscr{S}(A).$$

Because of the facts that

$$\mathscr{H}(P(\omega,\tau)) = 2\mathscr{H}(M(\omega,\tau)) - \mathscr{H}(A),$$

$\mathscr{H}(P(\omega,\tau))$ is Hermitian positive definite and $\mathscr{H}(A)$ is Hermitian positive semidefinite, we see that $\mathscr{H}(M(\omega,\tau))$ is Hermitian positive definite. In the light of Corollary 3.4 we immediately know that the splitting (3.7) is a convergent splitting.

Evidently, the conditions in Example 3.1 for guaranteeing the convergence of the matrix splitting (3.7) is less restrictive than those in [23]. As a matter of fact, here, we allow the situation that the original coefficient matrix A of the system of linear equations (1.1) is semidefinite.

4. Application to the generalized saddle-point problems

Consider the generalized saddle-point problem

$$Ax := \begin{pmatrix} W & C^{\mathrm{T}} \\ -C & Z \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} := b, \tag{4.1}$$

where $W \in \mathbb{R}^{\ell\times\ell}$, $C \in \mathbb{R}^{m\times\ell}$, $Z \in \mathbb{R}^{m\times m}$, $f \in \mathbb{R}^{\ell}$, $g \in \mathbb{R}^{m}$, and $m \leqslant \ell$.

This class of linear systems arises in many scientific and engineering applications such as mixed finite element approximation of elliptic partial differential equations, optimization, optimal control, structural analysis and electrical networks, see [20].

Let $D_1 \in \mathbb{R}^{\ell\times\ell}$ and $D_2 \in \mathbb{R}^{m\times m}$ be symmetric positive definite matrices, and $\widehat{L} \in \mathbb{R}^{\ell\times\ell}$ be the strictly lower triangular part of $\mathscr{S}(W)$. Then the block two-by-two matrices

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \quad L = \begin{pmatrix} -\widehat{L} & 0 \\ C & 0 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} \widehat{L}^{\mathrm{T}} - \mathscr{H}(W) + D_1 & -C^{\mathrm{T}} \\ 0 & -Z + D_2 \end{pmatrix}$$

satisfy

$$A = D - L - U.$$

Given two positive parameters γ and ω, let us define the matrices

$$\begin{cases} M(\gamma,\omega) = \frac{1}{\omega}(D - \gamma L), \\ N(\gamma,\omega) = \frac{1}{\omega}[(1-\omega)D + (\omega-\gamma)L + \omega U]. \end{cases}$$

Then $M(\gamma,\omega)$ is non-singular and

$$A = M(\gamma,\omega) - N(\gamma,\omega)$$

forms a splitting of the matrix $A \in \mathbb{R}^{(\ell+m)\times(\ell+m)}$ in (4.1). We call this splitting a *modified accelerated overrelaxation* (MAOR) splitting; see [3,4]. Correspondingly, this splitting leads to the following MAOR iteration method for the generalized saddle-point problem (4.1):

$$\begin{pmatrix} u^{(k+1)} \\ p^{(k+1)} \end{pmatrix} = T(\gamma,\omega) \begin{pmatrix} u^{(k)} \\ p^{(k)} \end{pmatrix} + \omega(D-\gamma L)^{-1} \begin{pmatrix} f \\ g \end{pmatrix}, \tag{4.2}$$

where

$$T(\gamma,\omega) := M(\gamma,\omega)^{-1}N(\gamma,\omega) = (D-\gamma L)^{-1}[(1-\omega)D + (\omega-\gamma)L + \omega U]$$

is the MAOR iteration matrix.

Obviously, when $W \in \mathbb{R}^{\ell\times\ell}$ is symmetric positive definite and $\gamma = \omega$, the MAOR iteration method (4.2) reduces to the inexact Uzawa method, and if, in particular, $D_1 = W$ and $D_2 = I$, it reduces to the Uzawa method, see [17,7].

By making use of Corollary 3.4, we can obtain the following convergence theorem for the MAOR iteration method (4.2).

Theorem 4.1. *Let $A \in \mathbb{R}^{(\ell+m)\times(\ell+m)}$ be the saddle-point matrix defined as in (4.1). Assume that $\mathscr{H}(W) \in \mathbb{R}^{\ell\times\ell}$ is positive semidefinite, $C \in \mathbb{R}^{m\times\ell}$ has full row rank, $Z \in \mathbb{R}^{m\times m}$ is symmetric positive semidefinite, and $\mathrm{null}(\mathscr{H}(W)) \cap \mathrm{null}(C) = \{0\}$. Here, we denote by $\mathrm{null}(\cdot)$ the null space of the corresponding matrix. Assume that $D_1 \in \mathbb{R}^{\ell\times\ell}$ and $D_2 \in \mathbb{R}^{m\times m}$ are symmetric positive definite matrices. Then the MAOR iteration method (4.2) is convergent, i.e., $\rho(T(\gamma,\omega)) < 1$, provided that for some $\nu > 1$ the parameters ω and γ satisfy*

$$\gamma = \nu\omega, \quad \omega > 0 \quad \textit{and} \quad \beta(\nu)\omega < 2\alpha, \tag{4.3}$$

where

$$\alpha = \min\{\lambda_{\min}(D_1), \lambda_{\min}(D_2)\}, \quad \beta(\nu) = \lambda_{\max}(V(\nu)),$$

with

$$V(\nu) = \begin{pmatrix} \mathscr{H}(W) - \nu(\widehat{L} + \widehat{L}^{\mathrm{T}}) & \nu C^{\mathrm{T}} \\ \nu C & Z \end{pmatrix}$$

and $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ being the smallest and the largest eigenvalues of the corresponding matrix.

Proof. Under the given assumptions we know that the matrix $A \in \mathbb{R}^{(\ell+m)\times(\ell+m)}$ is non-singular and positive semidefinite. Let $P = 2M - A$. Then by straightforward computations we have

$$\begin{cases} \mathscr{H}(M) = \frac{1}{\omega}D - \frac{\gamma}{2\omega}(L + L^{\mathrm{T}}) = \frac{1}{\omega}D - \frac{\nu}{2}(L + L^{\mathrm{T}}), \\ \mathscr{S}(M) = -\frac{\gamma}{2\omega}(L - L^{\mathrm{T}}) = \frac{\nu}{2}\mathscr{S}(A) \end{cases}$$

and

$$\begin{cases} \mathscr{H}(P) = 2\mathscr{H}(M) - \mathscr{H}(A) = \frac{2}{\omega}D - V(\nu), \\ \mathscr{S}(P) = 2\mathscr{S}(M) - \mathscr{S}(A) = (\nu - 1)\mathscr{S}(A). \end{cases}$$

It then follows that $\mathscr{H}(P)$ is symmetric positive definite, being equivalent to the positive definiteness of P, when the parameters ω and γ are within the domain defined by (4.3). By making use of Corollary 3.4 we immediately obtain $\rho(T(\gamma,\omega)) < 1$. $\square$

5. Application to the two-stage iteration methods

In this section, we use Theorem 3.1 to derive sufficient convergence conditions for both pointwise and blockwise two-stage iteration methods for solving the system of linear equations (1.1).

5.1. Convergence of the two-stage iteration method

Let $A = M - N$ be a splitting of the matrix $A \in \mathbb{C}^{n\times n}$ and $M = F - G$ be a splitting of the matrix $M \in \mathbb{C}^{n\times n}$. Then the two-stage iteration method for solving the system of linear equations (1.1) can be described as follows:

Method 5.1 (*The two–stage iteration method* [25, 26, 16]). Given an initial vector $x^{(0)} \in \mathbb{C}^n$ and a positive integer p.

For $k = 1, 2, \ldots,$
  Set $y^{(0)} = x^{(k-1)}$.

For $j = 1, 2, \ldots, p$,
Solve $Fy^{(j)} = Gy^{(j-1)} + Nx^{(k-1)} + b$.
Set $x^{(k)} = y^{(p)}$.

Method 5.1 can be rewritten in the matrix–vector form:

$$x^{(k)} = (F^{-1}G)^p x^{(k-1)} + \sum_{j=0}^{p-1} (F^{-1}G)^j F^{-1}(Nx^{(k-1)} + b). \tag{5.1}$$

Let $H = F^{-1}G$. If 1 is not an eigenvalue of either H or H^p, then by making use of the identity

$$\sum_{j=0}^{p-1} H^j(I - H) = I - H^p,$$

we can equivalently express (5.1) as

$$x^{(k)} = H^p x^{(k-1)} + (I - H^p)(I - H)^{-1}F^{-1}(Nx^{(k-1)} + b).$$

Denote by

$$T_p = H^p + (I - H^p)(I - H)^{-1}F^{-1}N$$

the iteration matrix of Method 5.1. Because

$$\begin{aligned} I - T_p &= (I - H^p)(I - (I - H)^{-1}F^{-1}N) \\ &= (I - H^p)(I - H)^{-1}(I - F^{-1}G - F^{-1}N) \\ &= (I - H^p)(I - H)^{-1}F^{-1}A \end{aligned}$$

and A is non-singular, it is obvious that $I - T_p$ is non-singular if and only if 1 is not an eigenvalue of either H or H^p. Therefore, we know that there exists a unique pair of matrices M_{T_p} and N_{T_p} such that

$$A = M_{T_p} - N_{T_p} \quad \text{and} \quad T_p = M_{T_p}^{-1}N_{T_p}.$$

Such matrices M_{T_p} and N_{T_p} are defined by

$$M_{T_p} = F(I - H)(I - H^p)^{-1} \quad \text{and} \quad N_{T_p} = M_{T_p} - A. \tag{5.2}$$

Based on the above preparation, we can establish the following convergence theorems for Method 5.1 when the coefficient matrix $A \in \mathbb{C}^{n\times n}$ of the linear system (1.1) is positive definite, but not necessarily Hermitian.

Theorem 5.1. *Let $A \in \mathbb{C}^{n\times n}$ be a positive definite matrix, $A = M - N$ a generalized P-regular splitting of the matrix A, $M \in \mathbb{C}^{n\times n}$ a Hermitian positive definite matrix, and $M = F - G$ a Hermitian and convergent splitting of the matrix M. Then, for any even positive integer p, $A = M_{T_p} - N_{T_p}$, with M_{T_p} and N_{T_p} being defined by (5.2), is a generalized P-regular splitting. Hence, by Theorem 3.1, Method 5.1 converges to the unique solution of the system of linear equations (1.1).*

Proof. Because $M = F - G$ is a convergent splitting, it holds that $\rho(H) := \rho(F^{-1}G) < 1$. Hence, $\rho(H^p) = \rho(H)^p < 1$. It then follows that both matrices $I - H$ and $I - H^p$ are non-singular. As

$$\begin{aligned}M_{T_p} &= F(I-H)(I-H^p)^{-1}\\ &= F(I-F^{-1}G)(I-(F^{-1}G)^p)^{-1}\\ &= M(I-H^p)^{-1}\\ &= M\sum_{j=0}^{+\infty} H^{pj}\end{aligned}$$

by using (5.2) we have

$$\begin{aligned}M_{T_p}+N_{T_p} &= 2M(I-H^p)^{-1}-A\\ &= 2M((I-H^p)^{-1}-I)+M+N\\ &= 2M\sum_{j=1}^{+\infty} H^{pj}+M+N.\end{aligned} \tag{5.3}$$

In addition, for any even positive integer p, we have

$$MH^{pj}=F(I-H)H^{pj}=FH^{\frac{pj}{2}}(I-H)H^{\frac{pj}{2}}=(GF^{-1})^{\frac{pj}{2}}M(F^{-1}G)^{\frac{pj}{2}}. \tag{5.4}$$

Since $M=F-G$ is a Hermitian splitting, we know that M, F and G are Hermitian matrices. Moreover, as M is Hermitian positive definite, (5.4) shows that MH^{pj} are Hermitian positive semidefinite matrices for all positive integers j. Therefore, M_{T_p} is a Hermitian positive definite matrix. From (5.3) we see that the Hermitian part of $2M_{T_p}-A$ is exactly the Hermitian part of $2M-A$ plus the Hermitian positive semidefinite matrix $2\sum_{j=1}^{+\infty}MH^{pj}$, i.e.,

$$\mathscr{H}(2M_{T_p}-A)=\mathscr{H}(2M-A)+2\sum_{j=1}^{+\infty}MH^{pj},$$

and the skew-Hermitian part of $2M_{T_p}-A$ is exactly the skew-Hermitian part of $2M-A$, i.e.,

$$\mathscr{S}(2M_{T_p}-A)=\mathscr{S}(2M-A).$$

Because $A\in\mathbb{C}^{n\times n}$ is a positive definite matrix and $A=M-N$ is a generalized P-regular splitting, we know that $A=M_{T_p}-N_{T_p}$ is a generalized P-regular splitting, too. Now, by Theorem 3.1 we immediately know that $\rho(T_p)<1$. $\square$

If the number p of the inner iteration steps is *any* positive integer, we can also prove the convergence of Method 5.1 when the matrix G is required to be Hermitian positive semidefinite.

Theorem 5.2. *Let $A\in\mathbb{C}^{n\times n}$ be a positive definite matrix, $A=M-N$ a generalized P-regular splitting of the matrix A, $M\in\mathbb{C}^{n\times n}$ a Hermitian positive definite matrix, and $M=F-G$ a Hermitian splitting of the matrix M such that $G\in\mathbb{C}^{n\times n}$ is Hermitian positive semidefinite. Then, for any positive integer p, $A=M_{T_p}-N_{T_p}$, with M_{T_p} and N_{T_p} being defined by (5.2), is a generalized P-regular splitting, and Method 5.1 converges to the unique solution of the system of linear equations (1.1).*

Proof. Since $M\in\mathbb{C}^{n\times n}$ is Hermitian positive definite and $G\in\mathbb{C}^{n\times n}$ is Hermitian positive semidefinite, the matrices $F=M+G$ and $F+G=M+2G$ are Hermitian positive definite. Therefore, $M=F-G$ is a Hermitian P-regular splitting. It then follows that $\rho(H):=\rho(F^{-1}G)<1$

and, hence, $I - H^p$ is non-singular for any positive integer p. This shows that M_{T_p} is a well-defined Hermitian matrix.

Now, we further investigate the Hermitian positive semidefiniteness of the matrices MH^{pj}, $j = 1, 2, \ldots$. In fact, if pj is even, then by (5.4) we immediately see that MH^{pj} is Hermitian positive semidefinite. On the other hand, if pj is odd, we have

$$\begin{aligned} MH^{pj} &= (F - G)(F^{-1}G)^{pj} \\ &= G(F^{-1}G)^{pj-1} - G(F^{-1}G)^{pj} \\ &= (GF^{-1})^{\frac{pj-1}{2}} G(F^{-1}G)^{\frac{pj-1}{2}} - (GF^{-1})^{\frac{pj-1}{2}} GF^{-1}G(F^{-1}G)^{\frac{pj-1}{2}} \\ &= (GF^{-1})^{\frac{pj-1}{2}} (G - GF^{-1}G)(F^{-1}G)^{\frac{pj-1}{2}}. \end{aligned}$$

Noticing that the matrix

$$G - GF^{-1}G = G^{\frac{1}{2}}\left(I - G^{\frac{1}{2}}F^{-1}G^{\frac{1}{2}}\right)G^{\frac{1}{2}}$$

is Hermitian positive semidefinite due to the Hermitian positive semidefiniteness of the matrix $G^{\frac{1}{2}}F^{-1}G^{\frac{1}{2}}$ and $\rho\left(G^{\frac{1}{2}}F^{-1}G^{\frac{1}{2}}\right) = \rho(F^{-1}G) < 1$, we know that MH^{pj} is a Hermitian positive semidefinite matrix.

Therefore, for any positive integer p, M_{T_p} is a Hermitian positive definite matrix. From (5.3) we see that

$$\mathscr{H}(2M_{T_p} - A) = \mathscr{H}(2M - A) + 2\sum_{j=1}^{+\infty} MH^{pj}$$

and

$$\mathscr{S}(2M_{T_p} - A) = \mathscr{S}(2M - A).$$

Because $A \in \mathbb{C}^{n\times n}$ is a positive definite matrix and $A = M - N$ is a generalized P-regular splitting, we know that $A = M_{T_p} - N_{T_p}$ is a generalized P-regular splitting, too. Now, by Theorem 3.1 we immediately know that $\rho(T_p) < 1$. $\square$

5.2. *Convergence of the block two-stage iteration method*

We now consider the case that the coefficient matrix $A \in \mathbb{C}^{n\times n}$ of the linear system (1.1) is of the block form

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1q} \\ A_{21} & A_{22} & \cdots & A_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ A_{q1} & A_{q2} & \cdots & A_{qq} \end{pmatrix},$$

where $q(\leqslant n)$ is a given positive integer, $A_{ii} \in \mathbb{C}^{n_i\times n_i}$ are n_i-by-n_i submatrices, and $n_i (i = 1, 2, \ldots, q)$ are positive integers satisfying $\sum_{i=1}^{q} n_i = n$.

Let $M_i \in \mathbb{C}^{n_i\times n_i}$ $(i = 1, 2, \ldots, q)$ be non-singular matrices and

$$M = \mathrm{Diag}(M_1, M_2, \ldots, M_q) \tag{5.5}$$

be the block diagonal matrix. Define $N = M - A$. Then $A = M - N$ forms a block splitting of the block matrix $A \in \mathbb{C}^{n\times n}$. In addition, let $M_i = F_i - G_i$ be a splitting of the matrix $M_i \in \mathbb{C}^{n_i\times n_i}$, $i = 1, 2, \ldots, q$, and denote by

$$F = \mathrm{Diag}(F_1, F_2, \ldots, F_q) \quad \text{and} \quad G = \mathrm{Diag}(G_1, G_2, \ldots, G_q). \tag{5.6}$$

Then $M = F - G$ forms a block splitting of the block matrix $M \in \mathbb{C}^{n\times n}$. We use z_i or $[z]_i$ to denote the i-th block entry of the block vector $z \in \mathbb{C}^n$, where z_i or $[z]_i \in \mathbb{C}^{n_i}$. Then, analogously to Method 5.1, we can establish the following block two-stage iteration method for solving the system of linear equations (1.1).

Method 5.2 (*The block two–stage iteration method* [25, 26, 16]). Given an initial vector $x^{(0)} \in \mathbb{C}^n$, a positive integer q, and positive integer sequences $\{p_{k,i}\}_{k=1}^{+\infty}$ $(i = 1, 2, \ldots, q)$.

For $k = 1, 2, \ldots,$
  For $i = 1, 2, \ldots, q,$
    Set $y_i^{(0)} = x_i^{(k-1)}$.
    For $j = 1, 2, \ldots, p_{k,i},$
      Solve $F_i y_i^{(j)} = G_i y_i^{(j-1)} + [Nx^{(k-1)} + b]_i$.
    Set $x_i^{(k)} = y_i^{(p_{k,i})}$.

Method 5.2 can be rewritten in the matrix–vector form:

$$x_i^{(k)} = (F_i^{-1}G_i)^{p_{k,i}} x_i^{(k-1)} + \sum_{j=0}^{p_{k,i}-1} (F_i^{-1}G_i)^j F_i^{-1}[Nx^{(k-1)} + b]_i, \quad i = 1, 2, \ldots, q. \tag{5.7}$$

Evidently, if $q = 1$ and $p_{k,i} = p$ for all k, then Method 5.2 directly reduces to Method 5.1.

For simplicity, in the sequel we assume that $p_{k,i} = p_i$ hold for all k. For $i = 1, 2, \ldots, q$, let $H_i = F_i^{-1}G_i$. Obviously, if 1 is not an eigenvalue of the matrices H_i and $H_i^{p_i}$, then the matrices $B_i = M_i(I - H_i^{p_i})^{-1}$ are well-defined and non-singular. Denote by

$$M_{T_p} = \mathrm{Diag}(B_1, B_2, \ldots, B_q). \tag{5.8}$$

Then the iteration matrix of Method 5.2 is

$$T_p := I - M_{T_p}^{-1}A.$$

Noticing that $I - T_p = M_{T_p}^{-1}A$ is non-singular, we know that

$$A = M_{T_p} - N_{T_p}$$

is the unique splitting induced by the matrix T_p, where

$$N_{T_p} = \mathrm{Diag}(B_1H_1^{p_1}, B_2H_2^{p_2}, \ldots, B_qH_q^{p_q}) + N. \tag{5.9}$$

Now, exactly following the demonstrations of Theorems 5.1 and 5.2, we can establish the following convergence theorems for Method 5.2 when the sequences $\{p_{k,i}\}_{k=1}^{+\infty}$ $(i = 1, 2, \ldots, q)$ of the inner iteration steps satisfy $p_{k,i} = p_i$ $(k = 1, 2, \ldots)$.

Theorem 5.3. *Let $A \in \mathbb{C}^{n\times n}$ be a positive definite matrix, $A = M - N$ a generalized P-regular splitting of the matrix A, $M \in \mathbb{C}^{n\times n}$ a Hermitian positive definite matrix, $N \in \mathbb{C}^{n\times n}$ a positive semidefinite matrix, and $M = F - G$ a Hermitian and convergent splitting of the matrix M, with M, F and G being defined in* (5.5) *and* (5.6). *Then, for any even positive integers p_i $(i = 1, 2, \ldots, q)$, $A = M_{T_p} - N_{T_p}$, with M_{T_p} and N_{T_p} being defined in* (5.8) *and* (5.9), *is a generalized P-regular splitting. Hence, by Theorem* 3.1, *Method* 5.2 *converges to the unique solution of the system of linear equations* (1.1).

Theorem 5.4. *Let $A \in \mathbb{C}^{n\times n}$ be a positive definite matrix, $A = M - N$ a generalized P-regular splitting of the matrix A, $M \in \mathbb{C}^{n\times n}$ a Hermitian positive definite matrix, and $M = F - G$ a Hermitian splitting of the matrix M such that $G \in \mathbb{C}^{n\times n}$ is Hermitian positive semidefinite, with M, F and G being defined in* (5.5) *and* (5.6). *Then, for any positive integers p_i $(i = 1, 2, \ldots, q)$, $A = M_{T_p} - N_{T_p}$, with M_{T_p} and N_{T_p} being defined in* (5.5) *and* (5.6), *is a generalized P-regular splitting, and Method* 5.2 *converges to the unique solution of the system of linear equations* (1.1).

In the following, we use two examples to further illustrate the conditions and examine the correctness of the above theorems.

Example 5.1. We consider the linear system (1.1) with the coefficient matrix $A \in \mathbb{C}^{6\times 6}$:

$$A = \begin{pmatrix} 6.2488 - 0.8744\imath & 0.2816 - 0.1408\imath & 0.232 + 0.884\imath & -0.876 - 0.312\imath & 0 & 0 \\ 0.2816 - 0.1408\imath & 4.7512 - 0.1256\imath & -0.276 + 0.888\imath & 0.568 + 0.716\imath & 0 & 0 \\ 0.2320 + 0.8840\imath & -0.2760 + 0.8880\imath & 5.980 - 0.740\imath & -0.640 + 0.320\imath & 0 & 0 \\ -0.8760 - 0.3120\imath & 0.5680 + 0.7160\imath & -0.640 + 0.320\imath & 5.020 - 0.260\imath & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 + 2\imath & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix},$$

where $\imath$ denotes the imaginary unit.

Let

$$M = \begin{pmatrix} 6.8744 & 0.1408 & 0.7160 & -0.8880 & 0 & 0 \\ 0.1408 & 6.1256 & 0.3120 & 0.8840 & 0 & 0 \\ 0.7160 & 0.3120 & 6.7400 & -0.3200 & 0 & 0 \\ -0.8880 & 0.8840 & -0.3200 & 6.2600 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

and

$$F = \begin{pmatrix} 8.2112 & -0.5616 & 1.5680 & -0.8240 & 0 & 0 \\ -0.5616 & 7.7888 & 0.3760 & 0.0320 & 0 & 0 \\ 1.5680 & 0.3760 & 7.5200 & -0.3600 & 0 & 0 \\ -0.8240 & 0.0320 & -0.3600 & 8.4800 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix}.$$

Then by defining

$$N = M - A \quad \text{and} \quad G = F - M,$$

we obtain the two-stage splitting $A = M - N$ of the matrix $A \in \mathbb{C}^{6\times 6}$, with $M = F - G$.

By straightforward computations, we obtain

$$\sigma(\mathscr{H}(A)) = \{4, 5, 6, 6, 7, 8\}, \quad \sigma(\mathscr{H}(M+N)) = \{2, 6, 6, 9, 9, 12\},$$
$$\sigma(\mathscr{H}(G)) = \{0, 0, 1, 2, 2, 3\}$$

and $\rho(\mathscr{S}(A)) = 2$. Therefore, $A \in \mathbb{C}^{6\times 6}$ is a positive definite matrix, $M \in \mathbb{C}^{6\times 6}$ is a Hermitian positive definite matrix, $A = M - N$ is a generalized P-regular splitting of the matrix $A \in \mathbb{C}^{6\times 6}$, $M = F - G$ is a Hermitian splitting, and $G \in \mathbb{C}^{6\times 6}$ is a Hermitian positive semidefinite matrix. By Theorem 5.2, we see that, for any positive integer p, Method 5.1 is convergent.

In fact, some computations yield

$$\rho(T_p) = 0.5154, 0.3566, 0.4444, 0.4815, \quad \text{for } p = 1, 2, 3, 4,$$

respectively.

Example 5.2. We consider the linear system (1.1) with the coefficient matrix $A \in \mathbb{R}^{6\times 6}$:

$$A = \begin{pmatrix} 4.7488 & 0.7816 & -0.4680 & -0.9760 & 0.1000 & -0.7000 \\ -0.2184 & 3.2512 & -0.1760 & -0.1320 & 0.7000 & 0.1000 \\ 0.1320 & -0.9760 & 4.4800 & -0.1400 & 0.5000 & -0.5000 \\ -0.1760 & 0.4680 & -1.1400 & 3.5200 & 0.5000 & 0.5000 \\ -0.1000 & -0.7000 & -0.5000 & -0.5000 & 5.0000 & 0.0000 \\ 0.7000 & -0.1000 & 0.5000 & -0.5000 & 0.0000 & 3.0000 \end{pmatrix}.$$

Let

$$M = \begin{pmatrix} 4.7488 & 0.2816 & -0.1680 & -0.5760 & 0 & 0 \\ 0.2816 & 3.2512 & -0.5760 & 0.1680 & 0 & 0 \\ -0.1680 & -0.5760 & 4.4800 & -0.6400 & 0 & 0 \\ -0.5760 & 0.1680 & -0.6400 & 3.5200 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

and

$$F = \begin{pmatrix} 6.3444 & -0.0692 & 0.9660 & -1.3380 & 0 & 0 \\ -0.0692 & 4.1556 & -0.4380 & 0.2340 & 0 & 0 \\ 0.9660 & -0.4380 & 5.4900 & -1.0700 & 0 & 0 \\ -1.3380 & 0.2340 & -1.0700 & 5.0100 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}.$$

Then by defining

$$N = M - A \quad \text{and} \quad G = F - M,$$

we obtain the two-stage splitting $A = M - N$ of the matrix $A \in \mathbb{R}^{6\times 6}$, with $M = F - G$.

By straightforward computations, we obtain

$$\sigma(\mathscr{H}(A)) = \{3, 3, 3, 5, 5, 5\}, \quad \sigma(\mathscr{H}(M+N)) = \{3, 3, 3, 5, 5, 5\},$$
$$\sigma(\mathscr{H}(G)) = \{0, 0, 1, 1, 2, 3\}$$

and $\rho(\mathscr{S}(A)) = 1$. Therefore, $A \in \mathbb{R}^{6\times 6}$ is a positive definite matrix, $M \in \mathbb{R}^{6\times 6}$ is a Hermitian positive definite matrix, $A = M - N$ is a generalized P-regular splitting of the matrix $A \in \mathbb{R}^{6\times 6}$,

$M = F - G$ is a Hermitian splitting, and $G \in \mathbb{R}^{6\times 6}$ is a Hermitian positive semidefinite matrix. By Theorem 5.2, we see that, for any positive integer p, Method 5.1 is convergent.

In fact, some computations yield

$$\rho(T_p) = 0.3273, 0.25, 0.2562, 0.2577, 0.2581, 0.2582, \quad \text{for } p = 1, 2, \ldots, 6,$$

respectively.

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